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Stochastic Resonance in Deterministic Chaotic Systems

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Abstract

We propose a mechanism which produces periodic variations of the degree of predictability in dynamical systems. It is shown that even in the absence of noise when the control parameter changes periodically in time, below and above the threshold for the onset of chaos, stochastic resonance effects appears. As a result one has an alternation of chaotic and regular, i.e. predictable, evolutions in an almost periodic way, so that the Lyapunov exponent is positive but some time correlations do not decay.

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The mechanism of stochastic resonance was initially introduced as a possible explanation of the long time climatic changes [1–4]. In the last years it has been used in a wide class of systems in physics and biology such as analogical circuits [5], ring laser [6], neurology [7,8], bistable systems [9,10], systems with colored noise [11], see Ref. [12] for a recent review.

The phenomenon can show up in bistable systems with a periodic forcing and a random perturbation. A typical example [1–4] is the evolution generated by the stochastic differential equation

$$\frac{dx}{dt} = -\frac{\partial V(x,t)}{\partial x} + \sqrt{2\sigma}\,\eta \tag{1}$$

where V is a time periodic double well potential

$$V(x,t) = \frac{x^4}{4} - \frac{x^2}{2} + A\,x\cos(\omega t), \tag{2}$$

and η white noise.

It can be shown [1–4] that there exists a range of values of A, σ and ω where the jumps between the two oscillating wells are strongly synchronized, as a consequence of a sort of resonance between periodic forcing and random perturbation.

Systems showing stochastic resonance are, in some sense, intermediate between regular and irregular ones, since they are described by a random process – the jumps do not follow a deterministic rule – which, nevertheless, exhibits a certain degree of regularity. For instance, the x time-correlation does not decay.

This letter shows that a similar behavior can arise in deterministic systems close to the onset of chaos when the control parameter varies periodically in time. Under appropriate conditions, the time evolution shows an alternation of regular and chaotic motion strongly synchronized with the time variation of the control parameter. The presence of deterministic chaos plays the role of the random perturbation, so that it would be more correct to speak of ‘chaotic resonance’ rather then of stochastic resonance. Chaotic resonance seems to be present in natural phenomena, such as the time evolution of weather, which is governed by a set of non-linear equations which surely exhibits deterministic chaos. Nevertheless, one

observes some elements of regularity such as high predictability during summer, at least in the tempered regions as the mediterranean countries, and very poor predictability during winter.

Although the alternation of seasons cannot be described by low dimensional systems, some qualitative features can be captured by toy models, which can be useful as a first step toward of the comprehension of the mechanism producing periodic variations of predictability in short and long term climate phenomena.

We have chosen to analyze the Lorenz model [13] which is the first geophysical dynamical system where deterministic chaos has been observed. We consider the original differential equations

$$\begin{cases} dx/dt = & 10\,(y-x) \\ dy/dt = & -x\,z + R(t)\,x - y \\ dz/dt = & x\,y - \frac{8}{3}\,z \end{cases} \tag{3}$$

where the control parameter has a periodic time variation:

$$R(t) = R_0 - A\,\cos(2\pi t/T). \tag{4}$$

The Lorenz model describes the convection of a fluid heated from below between two layers whose temperature difference is proportional to the Rayleigh number R. In our case, the periodic variations of R roughly mimic the seasonal changing on the solar heat inputs.

In order to get stochastic resonance effects without noise, the average Rayleigh number R_0 is assumed to be close to the threshold $R_{cr} = 24.74$ for the transition from stable fixed points to a chaotic attractor in the standard Lorenz model. The value of the amplitude A of the periodic forcing should be such that $R(t)$ oscillates below and above R_{cr}. For very large T, a good approximation of the solution is given by

$$x(t) = y(t) = \pm\sqrt{\frac{8}{3}(R(t)-1)} \qquad\qquad z(t) = R(t) - 1 \tag{5}$$

which is obtained by the fixed points of the standard Lorenz model by replacing R by $R(t)$. The stability of this solution is a rather complicated issue, which depends on the values of

R_0, A , and T. For instance, when $R_0 = 23.3$ and $A = 4$, we numerically found that the solution is stable for any value of T, although $R(t)$ can become larger than R_{cr}.

On the other hand, it is natural to expect that if R_0 is larger than R_{cr} the solution is unstable. In this case, for A large enough (at least $R_0 - A < R_{cr}$) one observes a mechanism similar to that of the stochastic resonance in bistable systems with random forcing. As in the case of the stochastic resonance we have a periodic variation in the dynamics (the control parameter) and the chaos plays the role of the noise. The value of T is crucial: for large T the systems behaves as follows. It is convenient to call

$$T_n \simeq nT/2 - T/4 \tag{6}$$

the times at which $R(t) = R_{cr}$. For $0 < t < T_1$, the control parameter $R(t)$ is smaller than R_{cr} so that the system is stable and the trajectory is close to one of the two solutions (5). For $T_1 < t < T_2$, one has $R(t) > R_{cr}$ and both solutions (5) are unstable so that the trajectory in a short time relaxes toward a sort of 'adiabatic' chaotic attractor. The chaotic attractor smoothly changes at varying R above the threshold R_{cr}, but if T is large enough, this dependence can be neglected in a first approximation. However, when $R(t)$ becomes again smaller than R_{cr}, the 'adiabatic' attractor disappears and, in general, the system is far from the stable solutions (5). But, since they are attracting, the system relaxes toward them. If the half-period T_1 is much larger than the relaxation time t_c, in general the system follows one of the two regular solutions (5) for $T_{2n+1} < t < T_{2n+2}$. However, there is a small but non-zero probability that the system has no enough time to relax to (5) and that its evolution remains chaotic. Figure 1 shows the time evolution for $T = 300$ (a) and $T = 1600$ (b). They provide a unambiguous numerical evidence that the jumps from the chaotic to the regular behavior (and the contrary) are well synchronized with $R(t)$, with probability close to 1 when the forcing period T is very long, as in Fig. 1b. On the other hand, for small value of T the system often does not perform the transition from the chaotic to the regular behavior, see Fig. 1a.

It is worth stressing that the system is chaotic. In both cases, in fact we found numerically

that the first Lyapunov exponent is positive, although the correlation function of the variable z does not decay. This is due to the presence of strong correlation between the regular intervals.

Figure 2 shows the probability distribution of the lengths of the irregular interval. One observes peaks around $T/2, 3T/2, 5T/2\cdots$, while the envelope of the probability distribution decreases exponentially. This feature can be easily explained. At $t = T_{2n}$ $(n = 1, 2\cdots)$ the system will be in some part of the 'adiabatic' chaotic attractor. The phase space is divided into two regions Ω_1 and Ω_2 such that if $\mathbf{x}(T_{2n})$ is contained in Ω_1 the trajectory during the following half-period will be very close to one of the two solutions (5). On the other hand the points $\mathbf{x}(T_{2n})$ contained in Ω_2 generate trajectories which remain far from (5). Calling π the measure of the region Ω_2 and noting that in the irregular intervals the correlations decay very fast, it follows that the probability, P_n, that the lengths of the irregular interval is close to T_{2n+1} is $P_n \simeq \widetilde{p}^n = \exp(-c\, n)$ with $c = -\ln \widetilde{p}$.

This feature has been observed in many other systems exhibiting stochastic resonance [7,8,14–16].

The probability of jumping a regular interval, π, decreases with the period of the forcing T, of course. Figure 3 shows that in the Lorenz model (3), the probability $P(T)$ to have an irregular interval longer than T decreases as:

$$P(T) = \int_T^\infty p(\tau)\, d\tau \simeq e^{-\alpha T} \tag{7}$$

where $p(\tau)$ is the probability distribution of the length of the irregular interval.

Without entering in the details, we briefly discuss the effect of a random forcing, of strength σ, in the case where $R(t) - R_{cr}$ changes sign during the time evolution but the solutions (5), in the absence of the noise, are stable. In practice, we consider the Langevin equation

$$\begin{cases} dx/dt = & 10\,(y - x) + \sqrt{2\sigma}\,\eta_1 \\ dy/dt = & -x\,z + R(t)\,x - y + \sqrt{2\sigma}\,\eta_2 \\ dz/dt = & x\,y - \frac{8}{3}\,z + \sqrt{2\sigma}\,\eta_3 \end{cases} \tag{8}$$

where $\eta_i(t)$ are uncorrelated white noises i.e. $< \eta_i(t)\eta_j(t') >= \delta_{ij}\delta(t-t')$.

The numerical study of the model (8) reveals a phenomenology very close to the original stochastic resonance [1–4]. For small values of σ one has the same qualitative behavior obtained at $\sigma = 0$, while for σ slightly larger than a critical value σ_{cr} one has an alternation of regular and irregular motions. Now the Lyapunov exponent, computed treating the noise as an usual time-dependent term, is negative, i.e. two trajectories, initially close, with the same realization of the random forcing do not separate but stick exponentially fast. We stress that the Lyapunov exponent computed in the above method is neither unique nor the most physically relevant characterization of the complexity of noisy systems [17].

It is not difficult to give a rough argument for the above features. In the time interval where $R(t) < R_{cr}$, because of the random noise, the distance δ between the state of the system $\mathbf{x}$ and the solutions (5) is $O(\sqrt{\sigma})$. During the half-period $T_{2n+1} < t < T_{2n+2}$, the typical distance δ grows exponentially:

$$\delta(t) \sim \sqrt{\sigma} e^{c(t-T_{2n+1})} \tag{9}$$

Very roughly, c is related to the largest real part of the eigenvalues of the stability matrix computed along the solutions (5). Calling L the size of the 'adiabatic attractor', if the strength of the random forcing is large enough, i.e.

$$\sigma > \sigma_{cr} \sim L^2 e^{-cT} \tag{10}$$

the system can jump into the 'adiabatic attractor' at a time between T_{2n+1} and T_{2n+2} and one has the same behavior shown in Fig. 1b.

This feature is quite similar to the original stochastic resonance, as the central role is played by the forcing term. Let us stress that the critical value σ_{cr} decreases very quickly with the period T.

In conclusion we have shown that the phenomenology of the stochastic resonance can appear in a dynamical system even in the absence of a random perturbation, when there is a periodic time variation of the control parameter around the onset of chaos. Instead of

the two minima in the double well potential considered by the original stochastic resonance, one has two dynamical states of the system: chaotic and regular. The role of the noise is played by the chaotic evolution itself. It is worth noting that one needs that the period T of the control parameter variations should be much larger than the internal relaxation time t_c toward the regular solution of the unperturbed system.

Stochastic resonance in chaotic systems has relevant consequence on the predictability problem. It shows that the predictability time is not trivially related to the Lyapunov exponent if T is large enough. During the regular intervals, one has an almost perfect predictability while in the irregular intervals the predictability time is given by the inverse of the Lyapunov exponent. Moreover, we have shown that there is a non-zero probability (vanishing when $T \to \infty$) to skip a regular interval. Using a pictorial language, we could say that the regular interval corresponds to the summer evolution, while the irregular one to winter. Although the Lorenz model is too naive for any attempt of a realistic description, it allows us to reproduce some important features of weather forecasting which motivated our work: the forecasting is limited up to a time proportional to the inverse Lyapunov exponent of the system during winter; there is a very high predictability in summer; there is a small but not negligible probability to have very bad summers (jumps of the regular intervals) where the weather is unpredictable.

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FIGURES

FIG. 1. Model with $A = 4$, $R_0 = 25.5$. z as a function of t/T for $T = 300$ (1a) and $T = 1600$ (1b).

FIG. 2. Model with $A = 4$, $R_0 = 25.5$. Probability density, p, to have an irregular interval $\tau = \Delta t/T$, for $T = 300$.

FIG. 3. Model with $A = 4$, $R_0 = 25.5$. Probability of jumping one or more regular intervals, P, as a function of the forcing period, T.